DYNAMIC PROBLEM OF THERMOELASTICITY FOR A HOLLOW CYLINDER

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We consider a dynamic problem of thermoelasticity for a hollow cylinder in which we assume the heat propagation speed to be finite.

We consider a long hollow cylinder with an inner radius of 1 and an outer radius equal to l, which is initially at zero temperature and to whose inner surface a constant temperature T_0 is suddenly applied. The end sections of the cylinder are kept stationary. The inner and outer surfaces of the cylinder are assumed to be stress free.

Thus we have a dynamic problem of thermoelasticity.

It was shown in [1] that for high temperature gradients in metals there is no classical correspondence between the heat flow and the gradient. Therefore to solve a dynamic problem of thermoelasticity it is necessary to employ a heat conduction equation which is hyperbolic, namely, one which takes into account the finite speed of heat propagation [1, 2]:

$$M^2 \frac{\partial^2 T}{\partial \mathrm{Fo}^2} + \frac{\partial T}{\partial \mathrm{Fo}} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial T}{\partial r} \tag{1}$$

with the following boundary and initial conditions:

$$T(1,\ \mathrm{Fo}) = T_0,\ T(l,\ \mathrm{Fo}) = 0;\ T(r,\ 0) = 0,\ \frac{\partial T(r,\ 0)}{\partial \mathrm{Fo}} = 0. \tag{2}$$

In Eqs. (1) and (2), with the exception of $T(r,\mathrm{Fo})$, all quantities are dimensionless.

Using the method of finite integral transforms we can write the solution of the problem (1)-(2) in the form [3]

$$T(r,\ \mathrm{Fo}) = T_0 - A \ln r + \sum_{n=1}^{\infty} \frac{W_n(\mathrm{Fo})}{N_n^2} V_0(\gamma_n,\ r), \tag{3}$$

where

$$V_0(\gamma_n,\ r) = A_n J_0(\gamma_n r) + Y_0(\gamma_n r);$$

$$A = \frac{T_0}{\ln l};\ A_n = -\frac{Y_0(\gamma_n)}{J_0(\gamma_n)},\ N_n^2 = \int_1^l V_0^2(\gamma_n,\ r)\, r\,dr;$$

$$W_n(\mathrm{Fo}) = B_n \left(e^{s_1 \mathrm{Fo}} - \frac{s_1}{s_2} e^{s_2 \mathrm{Fo}} \right);$$

$$s_{1,2} = \frac{-1 \pm \sqrt{1 - 4M^2\gamma_n^2}}{2M^2},$$

$$B_n = \frac{s_2 - s_1}{s_1} \int_1^l A(\ln r - T_0)\, V_0(\gamma_n,\ r)\, r\,dr;$$

and the γ_n are the roots of the characteristic equation

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$$J_0(\gamma_n)Y_0(\gamma_n l) - J_0(\gamma_n l)Y_0(\gamma_n) = 0.$$

We now determine the thermoelastic stresses in the cylinder. When the stressed state of a long cylinder, which is in a state of plane strain, is axially symmetric, there is no displacement in the direction of the angle φ and the relative elongation in the direction of the z axis can be taken to be constant. We take it equal to zero. The radial displacement u depends only on r and Fo, i.e., u = u(r, Fo). Hooke's Law is then expressed by the equations [4]

$$\begin{aligned} \sigma_r &= 2G\frac{1-\mu}{1-2\mu}\left[\frac{\partial u}{\partial r} + \frac{\mu}{1-\mu}\cdot\frac{u}{r} - mT\right], \\ \sigma_\varphi &= 2G\frac{1-\mu}{1-2\mu}\left[\frac{\mu}{1-\mu}\cdot\frac{\partial u}{\partial r} + \frac{u}{r} - mT\right], \\ \sigma_z &= 2G\frac{1-\mu}{1-2\mu}\left[\mu\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) - (1+\mu)\alpha T\right]. \end{aligned} \tag{4}$$

The equation of motion may be written as [4]:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\varphi}{r} = \rho\frac{\partial^2 u}{\partial \mathrm{Fo}^2}. \tag{5}$$

Substituting the Eqs. (4) into Eq. (5), we obtain

$$\frac{1}{c^2}\cdot\frac{\partial^2 u}{\partial \mathrm{Fo}^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial u}{\partial r} - \frac{u}{r^2} - m\frac{\partial T}{\partial r} \quad (1 < r < l). \tag{6}$$

Since the inner and outer surfaces of the cylinder are stress free, we see that $\sigma_r = 0$ on these surfaces. Therefore the boundary conditions for Eq. (6) are as follows:

$$\frac{\partial u}{\partial r} + \frac{\mu}{1-\mu}\cdot\frac{u}{r} - mT = 0 \quad \text{for} \quad r=1 \text{ and } r=l. \tag{7}$$

The initial conditions are of the form

$$u = \frac{\partial u}{\partial \mathrm{Fo}} = 0 \quad \text{for} \quad \mathrm{Fo} = 0. \tag{8}$$

We write the solution of Eq. (6) as the sum of a quasistatic term ψ(r, Fo) and a dynamic term θ(r, Fo):

$$u(r, \mathrm{Fo}) = \psi(r, \mathrm{Fo}) + \theta(r, \mathrm{Fo}). \tag{9}$$

The quasistatic term ψ(r, Fo) must satisfy the equation

$$\psi'' + \frac{1}{r}\psi' - \frac{1}{r^2}\psi - mT' = 0, \tag{10}$$

where the primes indicate differentiation with respect to r, subject to the boundary conditions (7) in which u(r, Fo) is to be replaced by ψ(r, Fo).

The solution of Eq. (10), satisfying the boundary conditions (7), has the form

$$\psi(r, \mathrm{Fo}) = \frac{m}{2}\left[(1-2\mu)r + \frac{1}{r}\right]\bar{T}(l, \mathrm{Fo}) + \frac{m}{r}\int_1^r T(\rho, \mathrm{Fo})\rho\, d\rho, \tag{11}$$

where

$$\bar{T}(r, \mathrm{Fo}) = \frac{2}{r^2-1}\int_1^r T(\rho, \mathrm{Fo})\rho\, d\rho$$

denotes the weighted mean temperature of the cylinder with inner radius r.

Substituting the solution (9) into Eq. (6), and taking Eq. (10) into account, we will have the following equation for determining the dynamic term $\theta(r,\mathrm{Fo})$:

$$\frac{1}{c^2}\cdot\frac{\partial^2\theta}{\partial\,\mathrm{Fo}^2}=\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial\theta}{\partial r}-\frac{\theta}{r^2}-\frac{1}{c^2}\cdot\frac{\partial^2\psi}{\partial\,\mathrm{Fo}^2}\quad(1<r<l). \tag{12}$$

The quasistatic term $\psi(r,\mathrm{Fo})$ satisfies the boundary conditions (7), therefore the dynamic term in the solution must satisfy the homogeneous boundary conditions

$$\frac{\partial\theta}{\partial r}+\frac{\mu}{1-\mu}\cdot\frac{\theta}{r}=0 \text{ for } r=1 \text{ and } r=l, \tag{13}$$

and the initial conditions

$$\theta=-\psi,\ \frac{\partial\theta}{\partial\,\mathrm{Fo}}=-\frac{\partial\psi}{\partial\,\mathrm{Fo}} \text{ for } \mathrm{Fo}=0. \tag{14}$$

Solving the problem (12)-(14) by the method of characteristic functions, we obtain

$$\theta(r,\ \mathrm{Fo})=\sum_{n=1}^{\infty}\bar{\theta}_n(\mathrm{Fo})\,W_n(r)\quad(1\leqslant r\leqslant l),$$

where

$$\bar{\theta}_n(\mathrm{Fo})=-\psi_n(\mathrm{Fo})+\gamma_n\int_0^{\mathrm{Fo}}\psi_n(\tau)\sin\gamma_n(\mathrm{Fo}-\tau)\,d\tau,$$

$$\psi_n(\mathrm{Fo})=\frac{1}{N_n^2c^2}\int_1^l\psi(r,\ \mathrm{Fo})\,W_n(r)\,rdr,$$

$$N_n^2=\frac{1}{c^2}\int_1^l W_n^2(r)\,rdr,\ W_n(r)=A_nJ_1(\gamma_n r)+Y_1(\gamma_n r),$$

$$A_n=-\frac{\dfrac{\gamma_n}{c}Y_0\left(\dfrac{\gamma_n}{c}\right)-\dfrac{1-2\mu}{1-\mu}Y_1\left(\dfrac{\gamma_n}{c}\right)}{\dfrac{\gamma_n}{c}J_0\left(\dfrac{\gamma_n}{c}\right)-\dfrac{1-2\mu}{1-\mu}J_1\left(\dfrac{\gamma_n}{c}\right)}. \tag{15}$$

The characteristic values γ_n (the frequencies of the free radial oscillations of the cylinder) are such that

$$\gamma_n=\frac{\pi n}{\delta}+\frac{\left(\frac{7}{8}-\frac{\mu}{1-\mu}\right)\frac{\delta}{l}}{\pi n}+\frac{\xi(n)}{n^2},\ \delta=l-1, \tag{16}$$

$\xi(n)$ is a bounded function for $n=1,2,\ldots$

Let us now calculate the quasistatic radial and tangential stresses. Substituting the expression for $\psi(r,\mathrm{Fo})$ from Eq. (11) into Eq. (4), we obtain expressions for the quasistatic stresses

$$\sigma_r^{st}=\frac{\alpha E}{2(1-\mu)}\left(1-\frac{1}{r^2}\right)\left[\bar{T}(l,\ \mathrm{Fo})-\bar{T}(r,\ \mathrm{Fo})\right],$$

$$\sigma_\varphi^{st}=\frac{\alpha E}{2(1-\mu)}\left[\left(1+\frac{1}{r^2}\right)\bar{T}(l,\ \mathrm{Fo})+\left(1-\frac{1}{r^2}\right)\bar{T}(r,\ \mathrm{Fo})-2T(r,\ \mathrm{Fo})\right]. \tag{17}$$

Let us determine σ_r^{st} as $\mathrm{Fo}\to 0$:

$$\lim_{\mathrm{Fo}\to 0}\sigma_r^{st}=\lim_{\mathrm{Fo}\to 0}\left\{\frac{\alpha E}{2(1-\mu)}\left(1-\frac{1}{r^2}\right)\left[\bar{T}(l,\ \mathrm{Fo})-\bar{T}(r,\ \mathrm{Fo})\right]\right\}=0,$$

since

$$\lim_{\mathrm{Fo}\to 0}\int_1^r T(r,\ \mathrm{Fo})\,rdr=0.$$

Evaluating σ_φ^{st} as Fo → 0, we shall have

$$\lim_{Fo\to 0}\sigma_\varphi^{st} = \begin{cases} -\dfrac{\alpha E T_0}{1-\mu} & \text{if} \quad r=1, \\ 0 & \text{if} \quad r>1. \end{cases} \tag{18}$$

Thus, as a consequence of the discontinuous nature of the given temperature change on the inner surface of the cylinder (r = 1), where the temperature jumps from zero to T_0, the tangential stress σ_φ^{st} has a discontinuity at r = 1 (a "stationary" jump). Consequently, there is a jump change in σ_φ^{st} from zero to $\alpha E T_0/1-\mu$.

However the radial stress σ_r^{st} is a continuous function of r.

Substituting the dynamic term in the displacement u(r,Fo) from Eq. (15) into the expressions (4) for the stresses, we obtain expressions for the dynamic stresses:

$$\sigma_r^d = \sum_{n=1}^{\infty}\theta_n(\mathrm{Fo})\left\{A_n\left[\frac{\gamma_n}{c}J_0\left(\gamma_n\frac{r}{c}\right)-\frac{1-2\mu}{r(1-\mu)}J_1\left(\gamma_n\frac{r}{c}\right)\right]\right.$$
$$\left.+\frac{\gamma_n}{c}Y_0\left(\gamma_n\frac{r}{c}\right)-\frac{1-2\mu}{r(1-\mu)}Y_1\left(\gamma_n\frac{r}{c}\right)\right\},$$

$$\sigma_\varphi^d = \sum_{n=1}^{\infty}\theta_n(\mathrm{Fo})\left\{A_n\left[\frac{\gamma_n}{c}\cdot\frac{\mu}{1-\mu}J_0\left(\gamma_n\frac{r}{c}\right)\right.\right.$$
$$\left.+\frac{1-2\mu}{r(1-\mu)}J_1\left(\gamma_n\frac{r}{c}\right)\right]+\frac{\gamma_n}{c}\cdot\frac{\mu}{1-\mu}Y_0\left(\gamma_n\frac{r}{c}\right) \tag{19}$$
$$\left.+\frac{1-2\mu}{r(1-\mu)}Y_1\left(\gamma_n\frac{r}{c}\right)\right\}.$$

The tangential stress σ_φ at r = 1, as a result of the condition $\sigma_r = 0$, may be determined from the relation

$$\sigma_\varphi = \frac{2G}{1-\mu}\left[u-(1+\mu)\alpha T_0\right] \quad \text{for} \quad r=1. \tag{20}$$

If we let Fo → 0, then u(1,Fo) → 0, and from Eq. (20) we obtain

$$\lim_{Fo\to 0}\sigma_\varphi = -\frac{\alpha E T_0}{1-\mu}. \tag{21}$$

From this and from Eq. (18) we see that the tangential stress σ_φ on the inner surface of the cylinder coincides immediately, after the instantaneous heating, with the quasistatic tangential stress.

The solution of the problem (12)-(14) for the dynamic part of the solution u(r,Fo) was written in the form of a series of the type

$$\theta(r,\ \mathrm{Fo}) = \sum_{n=1}^{\infty}\bar{\theta}_n(\mathrm{Fo})\,W_n(r) \quad (1\leqslant r\leqslant l), \tag{22}$$

where $\bar{\theta}_n$(Fo) and W_n(r) may be obtained from the expressions (15). However, the series (22) converges slowly for small values of Fo, i.e., immediately following the effect of the thermal shock. This is explained by the fact that the series (22) is known over the whole domain of variation for r ($1 \le r \le l$) whereas the deformations are of a local nature, i.e., the displacement θ(r,Fo) is different from zero only in the region $1 \le r \le 1 + c\mathrm{Fo}$, $0 < \mathrm{Fo} < (l-1)/c$, and is zero in the remaining part. In order to improve the convergence of the solution of the problem (12)-(14) we exclude from the domain of the expansion of the solution in a series of characteristic functions the undisturbed part, where the displacement θ(r,Fo) is equal to zero.

As a result, the thermal shock on the inner surface of the cylinder gives rise to an elastic cylindrical wave, which at the time instant Fo is located at the radius $1 + c\mathrm{Fo}$ $(0 < \mathrm{Fo} < (l-1)/c)$; moreover, at the front of the wave the displacement θ(r,Fo) must be equal to zero.

Taking this into account and also the fact that

$$\psi(r,\ 0)=\frac{\partial\psi(r,\ 0)}{\partial\,\mathrm{Fo}}=0,$$

we have a boundary-value problem for determining $\theta(r,\mathrm{Fo})$ for the values of the time $(0 < \mathrm{Fo} < (l-1)/c)$:

$$\frac{1}{c^2}\cdot\frac{\partial^2\theta}{\partial\,\mathrm{Fo}^2}=\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial\theta}{\partial r}-\frac{\theta}{r^2}-\frac{1}{c^2}\cdot\frac{\partial^2\psi}{\partial\,\mathrm{Fo}^2}$$

$$\left(1<r<L,\ 0<\mathrm{Fo}<\frac{l-1}{c}\right), \tag{23}$$

$$\frac{\partial\theta}{\partial r}+k\theta=0 \quad \text{for} \quad r=1,$$

$$\theta=0 \quad \text{for} \quad r=L,$$

$$\theta=\frac{\partial\theta}{\partial\,\mathrm{Fo}}=0 \quad \text{for} \quad \mathrm{Fo}=0, \tag{24}$$

where $k=\mu/1-\mu$; $L=1+c\mathrm{Fo}$.

If we should need to find the displacement $\theta(r,\mathrm{Fo})$ for the time interval $(0 < \mathrm{Fo} \leq \mathrm{Fo}^*)$, where $0<\mathrm{Fo}<(l-1)/c$, t is necessary to put $L=1+c\mathrm{Fo}^*$.

Solving the resulting problem by the method of characteristic functions, we obtain

$$\theta(r,\ \mathrm{Fo})=\sum_{n=1}^{\infty}\bar{\theta}_n(\mathrm{Fo})\,W_n^*(r) \quad (1\leqslant r\leqslant L,\ 0\leqslant\mathrm{Fo}\leqslant\mathrm{Fo}^*), \tag{25}$$

where

$$W_n^*(r)=Y_1\left(\gamma_n\frac{r}{c}\right)-\frac{Y_1\left(\gamma_n\frac{L}{c}\right)}{J_1\left(\gamma_n\frac{L}{c}\right)}J_1\left(\gamma_n\frac{r}{c}\right)$$

and $\bar{\theta}_n(\mathrm{Fo})$ is determined from Eqs. (15).

The characteristic values γ_n are such that

$$\gamma_n=\frac{\pi\left(n+\frac{1}{2}\right)}{\delta}+\frac{\frac{3}{8}\frac{\delta}{L}-k}{\pi\left(n+\frac{1}{2}\right)}+\frac{\xi(n)}{n^2}, \tag{26}$$

where $\delta=L-1$.

Thus the solution of the problem (12)-(14) is given by a series of the form (25), defined only for the disturbed (deformed) part of the cylinder $(1 \leq r \leq 1+c\mathrm{Fo};\ 0<\mathrm{Fo}<(l-1)/c)$.

The series (25) converges considerably faster than the series (22). Moreover, the terms of this series have a simpler form, thereby making the numerical computations easier.

Thus the solution of the problem (6)-(8) for $0<\mathrm{Fo}<(l-1)/c$, taking Eqs. (9) and (25) into account, is

$$u(r,\ \mathrm{Fo})=\psi(r,\ \mathrm{Fo})+\sum_{n=1}^{\infty}\bar{\theta}_n(\mathrm{Fo})\,W_n^*(r), \tag{27}$$

where $\psi(r,\mathrm{Fo})$ is determined by the expression (11) and represents the quasistatic part of the displacement $u(r,\mathrm{Fo})$. Representation of the solution of the problem (6)-(8) over the time interval $0<\mathrm{Fo}<(l-1)/c$ as a sum of a quasistatic part $\psi(r,\mathrm{Fo})$ and a dynamic part $\theta(r,\mathrm{Fo})$ may be explained by the fact that heat in the cylinder propagates with a speed c_q, which is less than the speed of propagation c of the dilatational waves in an elastic medium. Consequently, quasistatic stresses arise in the cylinder. Therefore, in order to obtain the solution of the problem (6)-(8) it is necessary to augment the solution (25) by a quasistatic term.

The elastic cylindrical wave reaches the outer surface of the cylinder at the time instant $\mathrm{Fo}=(l-1)/c$. The wave is then reflected. To find the displacement in the case of the reflected wave we can use the solution (15) for $\mathrm{Fo} > (l-1)/c$.

NOTATION

$M = (c/c_q)$	
$c = \sqrt{2(1-\mu)/(1-2\mu) \cdot G/\rho}$	is the dilatational wave speed in an elastic medium;
c_q	is the heat propagation speed;
$m = (1+\mu)\alpha/1-\mu$	
μ	is Poisson's ratio;
α	is the thermal coefficient for linear expansion;
ρ	is the density;
E	is Young's modulus;
G	is the shear modulus of elasticity;
$Fo = at/l^2$	is the Fourier number;
a	is the thermal diffusivity;
t	is the time;
$J_m(r)$, $Y_m(r)$,	are the Bessel functions of the first and second kinds of order m.

LITERATURE CITED

1. A. V. Lykov, Analytical Heat Diffusion Theory, Academic Press, New York (1968).
2. M. D. Mikhailov, Inzh. Fiz. Zh., 16, No. 1 (1969).
3. V. N. Gavdzinskii, in:Heat Conduction Research Studies [in Russian], Nauka i Tekhnika, Minsk (1967).
4. H. Parkus, Nonstationary Thermal Stresses [Russian translation], Fizmatgiz, Moscow (1963).